EVOLUTIONARY SYSTEM, GLOBAL ATTRACTOR, TRAJECTORY ATTRACTOR AND APPLICATIONS TO THE NONAUTONOMOUS REACTION-DIFFUSION SYSTEMS

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arXiv:1502.05074v2 [math.DS] 19 Jul 2015

ABSTRACT. In [Adv. Math., 267(2014), 277-306], Cheskidov and Lu develop a new framework of the evolutionary system that deals directly with the notion of a uniform global attractor due to Haraux, and by which a trajectory attractor is able to be defined for the original system under consideration. The notion of a trajectory attractor was previously established for a system without uniqueness by considering a family of auxiliary systems including the original one. In this paper, we further prove the existence of a notion of a strongly compact strong trajectory attractor if the system is asymptotically compact. As a consequence, we obtain the strong equicontinuity of all complete trajectories on global attractor and the finite strong uniform tracking property. Then we apply the theory to a general nonautonomous reaction-diffusion systems. In particular, we obtain the structure of uniform global attractors without any additional condition on nonlinearity other than those guarantee the existence of a uniform absorbing set. Finally, we construct some interesting examples of such nonlinearities. It is not known whether they can be handled by previous frameworks.

Keywords: evolutionary system, uniform global attractor, trajectory attractor, reaction-diffusion system, normal external force

Mathematics Subject Classification:35B40; 35B41; 35K57

1. INTRODUCTION

A mathematical object describing the long-time behavior of an autonomous infinite dimensional dissipative dynamical system with uniqueness is the global attractor, which is a minimal closed set that attracts all the trajectories starting from a bounded set in the phase space. The global attractor consists of points on complete bounded trajectories, that is, its structure is represented as a section at a fixed time of the kernel, a collection of all complete bounded trajectories of the system.

A natural generalization of the notion of a global attractor to the nonautonomous dynamical system is that of a uniform global attractor, initiated by Haraux [Ha91], defined additionally with the attracting uniformly with respect to (w.r.t.) the initial time. Chepyzhov and Vishik [CV94, CV02] put forward an auxiliary notion of a

S.L. was partially supported by Specialized Research Fund for the Doctoral Program of Higher Education (200805581025), NSFC 11001279 and the Fundamental Research Funds for the Central Universities (SYSU 11lgpy27).

uniform w.r.t. the symbol space attractor to study the structure of the uniform global attractor, based on the use of the so-called time symbol and constructing a symbol space as a suitable closure of the translation family of the symbol of the original system under consideration. However, the uniform w.r.t. the symbol space attractor does not always have to be identical to the original uniform global attractor (see [CL14]). On the other hand, the open problems in [Lu07, CL09] indicate that there may not exist such a required symbol space.

For the dynamical system without uniqueness, instead, the concept of a trajectory attractor, a global attractor in the trajectory space, was first introduced in [Se96] and further studied in [CV97, CV02, SY02]. Their method considers also a family of auxiliary systems containing the original one. In applications, the constructions use the phase spaces that are generally endowed with weaker convergence topologies than the natural ones. The sections of such trajectory attractors are defined as, for the autonomous case, the weak global attractors, or for nonautonomous case, the weak uniform w.r.t the symbol space attractors. Similarly, the Open Problem 6.7 in [CL14] shows that it is not clear whether these trajectory attractors satisfy the minimality property for the original nonautonomous system due to the nonuniqueness feature of the system.

Most recently, Cheskidov and Lu [CL14] developed, based on the previous studies [CF06, C09, CL09], a framework of evolutionary system to investigate the global attractors and the trajectory attractors for dissipative dynamical systems. Especially, for the nonautonomous case, the new approach deals directly with the notions of a uniform global attractor and a trajectory attractor and avoids the necessity of constructing a symbol space. Together with the advantage of a simultaneous use of weak and strong metrics, this method is applicable to arbitrary dissipative partial differential equation (PDE), no matter whether it is nonautonomous, or is no unique results of the corresponding Cauchy problem or does not possess a symbol space. In this paper, we further study a notion of a strongly compact strong trajectory attractor.

According to the theory in [CL14], an evolutionary system always processes a weak global attractor. Its structure is obtained via that of the weak global attractor for the closure of the evolutionary system under an assumption (see A1) satisfied by any dissipative PDE, since these two weak global attractors coincide. Moreover, a weak uniform tracking property is proved, which means that we can approximate in weak metric arbitrarily closely every trajectory of the evolutionary system for arbitrarily large time lengths by the trajectories on the weak global attractor after sufficiently long time. In particular, for the nonautonomous case, a trajectory attractor is naturally constructed for the originally considered system, rather than for a family of systems. The sections of the trajectory attractor are the weak global attractor.

The weak global attractor becomes a strongly compact strong global attractor, if the asymptotical compactness of an evolutionary system is provided. At the same time, the property of uniform tracking is valid in strong metric and trajectories converge strongly toward the trajectory attractor. In this paper, we continue investigating the properties of the evolutionary system, concerning on the trajectory attractor. More precisely, we prove the existence of a notion of a strongly compact strong trajectory attractor.

We prove that the trajectory attractor is compact in the space of continuous functions of time with values in the phase space endowed with the usual strong metric under consideration. As a consequence, we obtain a finite strong uniform tracking property that for any fixed accuracy and time length, a finite number of trajectories on the global attractor are able to capture in strong metric all trajectories after sufficiently large time. Moreover, all complete trajectories on global attractor is strongly equicontinuous. Notice again that such a strongly compact strong trajectory attractor is constructed for the original system we consider, and followed simultaneously with the strongly compact strong global attractor if the evolutionary system is asymptotically compact.

We apply the abstract theory to a dissipative reaction-diffusion system (RDS) that is a fundamental model in the theory of infinite dimensional dynamical systems. It is quite general that covers many examples arising in physics, chemistry and biology etc. We just list a few: the RDS with polynomial nonlinearity, Ginzburg-Landau equation, Chafee-Infante equation, Fitz-Hugh-Nagumo equations and Lotka-Volterra competition system. See e.g. [M87, T88, CV96, Ro01, CV02, SY02] for more. In the current paper, the RDS also serves as an example of a PDE that is nonautonomous, nonunique and lack of an appropriate symbol space.

The paper is organized as follows. In Section 2, we briefly recall the basic definitions of the theory of evolutionary system originally designed in [CF06, C09] for autonomous systems and developed in [CL09, CL14] especially for nonautonomous systems. Then, in Section 3, we show that the weak and strong uniform tracking properties are equivalent to the weak and strong trajectory attracting, respectively, under a natural condition. In particular, we define a strongly compact strong trajectory attractor as well as the finite strong uniform tracking property. Its existence and properties mentioned above are proved in Section 4. With the observations in Section 3 in mind, we present the results incorporating with the theory in [CL14], which will hint some point of view (see Remark 4.5).

In Section 5, we investigate the RDS with fixed time-dependent nonlinearity and driving force. The nonlinearity only satisfies the continuity, dissipativeness and growth conditions that do not guarantee the unique solvability and the assumption on the force is a translation boundedness condition, which is the weakest condition that ensures the existence of a bounded uniform absorbing ball. We take this ball as a phase space. The weak and strong metrics are metrics induced by the

usual weak and strong topologies respectively. We verify that the weak solutions of RDS form an evolutionary system satisfying A1. Therefore, we obtain the structure of the weak attractors (both the uniform global attractor and the trajectory attractor). In addition, if the force is normal then the evolutionary system is asymptotical compact. Hence, the two attractors are in fact strongly compact strong ones. The normality condition on the force was introduced in [LWZ05] and Lu [Lu06] by different ways. Now, it follows naturally from the energy inequality of a criterion of asymptotical compactness, known for the 3D Navier-Stokes equations [CF06] and formulated in [C09] and generalized to general cases in [CL09, CL14]. It is worth to mention that the time-dependence of the nonlinearity is not imposed on any additional assumption, for instance, which promises the existence of a symbol space. Hence, we give an answer to part of open problems in [Lu07] and [CL09].

In Section 6, for the reader's convenience, we first review concisely some results on the RDS obtained in previous works [CV02, Lu07, CL09], supposing additional conditions on the nonlinearity. Then, we recover them using our framework and discuss their relations to those obtained in Section 5. Finally, we construct several interesting examples of nonlinearities that only satisfy the continuity, dissipativeness and growth conditions. The extra assumptions imposed in [CV02, Lu07, CL09] do not hold any more. The pointwise limit function of one example as $t \to +\infty$ is discontinuous and the others have even no pointwise limit functions. These facts mean that the weak closure of the translation family of every example in the space of continuous functions of time with values in corresponding topological space is not complete. In other words, there does not exist a suitable symbol space required in previous studies.

2. Evolutionary system

Now we recall briefly the basic definitions on evolutionary systems (see [C09, CL14] for details). Assume that a set X is endowed with two metrics $\mathrm{d_s}(\cdot,\cdot)$ and $\mathrm{d_w}(\cdot,\cdot)$ respectively, satisfying the following conditions:

(1) X is $\mathrm{d_w}$-compact.
(2) If $\mathrm{d_s}(u_n, v_n) \to 0$ as $n \to \infty$ for some $u_n, v_n \in X$, then $\mathrm{d_w}(u_n, v_n) \to 0$ as $n \to \infty$.

Hence, we will refer to $\mathrm{d_s}$ as a strong metric and $\mathrm{d_w}$ as a weak metric. Let $\overline{A}^{\bullet}$ be the closure of a set $A \subset X$ in the topology generated by $\mathrm{d}_{\bullet}$. Here (the same below) $\bullet = \mathrm{s}$ or w. Note that any strongly compact ($\mathrm{d_s}$-compact) set is weakly compact ($\mathrm{d_w}$-compact), and any weakly closed set is strongly closed.

Let

$$\mathcal{T} := \{I : \ I = [T,\infty) \subset \mathbb{R}, \text{ or } I = (-\infty,\infty)\},$$

and for each $I \subset \mathcal{T}$, let $\mathcal{F}(I)$ denote the set of all X-valued functions on I. Now we define an evolutionary system $\mathcal{E}$ as follows.

Definition 2.1. *A map $\mathcal{E}$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subset \mathcal{F}(I)$ will be called an evolutionary system if the following conditions are satisfied:*

(1) $\mathcal{E}([0,\infty)) \neq \emptyset$.
(2) $\mathcal{E}(I+s) = \{u(\cdot) : \ u(\cdot + s) \in \mathcal{E}(I)\}$ *for all* $s \in \mathbb{R}$.
(3) $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}(I_1)\} \subset \mathcal{E}(I_2)$ *for all pairs* $I_1, I_2 \in \mathcal{T}$*, such that* $I_2 \subset I_1$.
(4) $\mathcal{E}((-\infty,\infty)) = \{u(\cdot) : \ u(\cdot)|_{[T,\infty)} \in \mathcal{E}([T,\infty)) \ \forall T \in \mathbb{R}\}$.

We will refer to $\mathcal{E}(I)$ as the set of all trajectories on the time interval I. $\mathcal{E}((-\infty,\infty))$ is called the kernel of $\mathcal{E}$ and its trajectories are called complete. Let $P(X)$ be the set of all subsets of X. For every $t \geq 0$, define a set-valued map

$$R(t) : P(X) \to P(X),$$
$$R(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}([0,\infty))\}, \qquad A \subset X.$$

Note that the assumptions on $\mathcal{E}$ imply that $R(t)$ enjoys the following property:

$$R(t+s)A \subset R(t)R(s)A, \qquad A \subset X, \quad t,s \geq 0.$$

Definition 2.2. *A set $\mathcal{A}_\bullet \subset X$ is a $\mathrm{d}_\bullet$-global attractor if $\mathcal{A}_\bullet$ is a minimal set that is*

(1) $\mathrm{d}_\bullet$*-closed.*
(2) $\mathrm{d}_\bullet$*-attracting: for any $B \subset X$ and $\epsilon > 0$, there exists t_0, such that*

$$R(t)B \subset B_\bullet(\mathcal{A}_\bullet, \epsilon) := \{y \in X : \inf_{x \in \mathcal{A}_\bullet} \mathrm{d}_\bullet(x,y) < \epsilon\}, \quad \forall\, t \geq t_0.$$

Definition 2.3. *The $\omega_\bullet$-limit of a set $A \subset X$ is*

$$\omega_\bullet(A) := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} R(t)A}^{\bullet}.$$

In order to extend the notion of invariance from a semiflow to an evolutionary system, we will need the following mapping:

$$\widetilde{R}(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}((-\infty,\infty))\}, \quad A \subset X, \ t \in \mathbb{R}.$$

Definition 2.4. *A set $A \subset X$ is positively invariant if*

$$\widetilde{R}(t)A \subset A, \qquad \forall t \geq 0.$$

A is invariant if

$$\widetilde{R}(t)A = A, \qquad \forall t \geq 0.$$

A is quasi-invariant if for every $a \in A$ there exists a complete trajectory $u \in \mathcal{E}((-\infty,\infty))$ with $u(0) = a$ and $u(t) \in A$ for all $t \in \mathbb{R}$.

Let Σ be a parameter set and $\{T(s) | s \geq 0\}$ be a family of operators acting on Σ satisfying $T(s)\Sigma = \Sigma$, $\forall s \geq 0$. Any element $\sigma \in \Sigma$ will be called (time) symbol and Σ will be called (time) symbol space. For instance, in many applications $\{T(s)\}$ is the translation semigroup and Σ is the translation family of the time-dependent items of the considered system or its closure in some appropriate topological space (for more examples see [CV02, CL14], the appendix in [CLR13]).

Definition 2.5. *A family of maps $\mathcal{E}_\sigma$, $\sigma \in \Sigma$ that for every $\sigma \in \Sigma$ associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}_\sigma(I) \subset \mathcal{F}(I)$ will be called a nonautonomous evolutionary system if the following conditions are satisfied:*

(1) $\mathcal{E}_\sigma([\tau,\infty)) \neq \emptyset, \forall\, \tau \in \mathbb{R}$.
(2) $\mathcal{E}_\sigma(I+s) = \{u(\cdot) : u(\cdot+s) \in \mathcal{E}_{T(s)\sigma}(I)\}, \forall s \geq 0$.
(3) $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}_\sigma(I_1)\} \subset \mathcal{E}_\sigma(I_2), \forall\, I_1, I_2 \in \mathcal{T}, I_2 \subset I_1$.
(4) $\mathcal{E}_\sigma((-\infty,\infty)) = \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \mathcal{E}_\sigma([\tau,\infty)), \forall\, \tau \in \mathbb{R}\}$.

It is shown in [CL09, CL14] that a nonautonomous evolutionary system can be reduced to an (autonomous) evolutionary system. Consequently, the above notions of invariance, quasi-invariance, and a global attractor are applicable. The global attractor in the nonautonomous case will be conventionally called a uniform global attractor (or simply a global attractor). However, for some evolutionary systems constructed from nonautonomous dynamical systems the associated symbol spaces are not known. See [CL14] and the following sections below for more details. Thus, we will not distinguish between autonomous and nonautonomous evolutionary systems. If it is necessary, we denote an evolutionary system with a symbol space Σ by $\mathcal{E}_\Sigma$ and its attractor by $\mathcal{A}^\Sigma$.

Definition 2.6. *Let $\mathcal{E}$ be an evolutionary system. If a map $\mathcal{E}^1$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}^1(I) \subset \mathcal{E}(I)$ is also an evolutionary system, we will call it an evolutionary subsystem of $\mathcal{E}$, and denote by $\mathcal{E}^1 \subset \mathcal{E}$.*

Definition 2.7. *An evolutionary system $\mathcal{E}_\Sigma$ is a system with uniqueness if for every $u_0 \in X$ and $\sigma \in \Sigma$, there is a unique trajectory $u \in \mathcal{E}_\sigma([0,\infty))$ such that $u(0) = u_0$.*

3. Uniform tracking property and trajectory attractor

An important property of a global attractor is that it captures a long-time behavior of every trajectory of an evolutionary system. In this section, we do some preparations for showing in the next section that this property can be formulated by the existence of a trajectory attractor.

Denote by $C([a,b];X_\bullet)$ the space of $\mathrm{d}_\bullet$-continuous X-valued functions on $[a,b]$ endowed with the metric

$$\mathrm{d}_{C([a,b];X_\bullet)}(u,v) := \sup_{t\in[a,b]} \mathrm{d}_\bullet(u(t),v(t)).$$

Let also $C([a,\infty);X_\bullet)$ be the space of $\mathrm{d}_\bullet$-continuous X-valued functions on $[a,\infty)$ endowed with the metric

$$\mathrm{d}_{C([a,\infty);X_\bullet)}(u,v) := \sum_{l\in\mathbb{N}} \frac{1}{2^l} \frac{\mathrm{d}_{C([a,a+l];X_\bullet)}(u,v)}{1+\mathrm{d}_{C([a,a+l];X_\bullet)}(u,v)}.$$

Note that the convergence in $C([a,\infty);X_\bullet)$ is equivalent to uniform convergence on compact sets.

Now we suppose that evolutionary systems $\mathcal{E}$ satisfy

$$\mathcal{E}([0,\infty)) \subset C([0,\infty);X_{\mathrm{w}}).$$

Definition 3.1. *A set $P \subset C([0,\infty);X_{\mathrm{w}})$ satisfies the weak uniform tracking property if for any $\epsilon > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*

$$\mathrm{d}_{C([t^*,\infty);X_{\mathrm{w}})}(u,v) < \epsilon, \tag{1}$$

for some trajectory $v \in P$. Furthermore, if there exists a finite subset $P^f \subset P$, such that (1) holds for some $v \in P^f$, then P satisfies the finite weak uniform tracking property.

Now we show that we may use the concept of trajectory attracting to state the weak uniform tracking property. Define the translation operators $T(s)$, $s \geq 0$,

$$(T(s)u)(\cdot) := u(\cdot + s)|_{[0,\infty)}, \quad u \in C([0,\infty);X_{\mathrm{w}}). \tag{2}$$

Due to the property 3 of the evolutionary system (see Definitions 2.1), we have that,

$$T(s)\mathcal{E}([0,\infty)) \subset \mathcal{E}([0,\infty)), \quad \forall\, s \geq 0.$$

Note that $\mathcal{E}([0,\infty))$ may not be closed in $C([0,\infty);X_{\mathrm{w}})$. We consider the dynamics of the translation semigroup $\{T(s)\}_{s\geq 0}$ acting on the phase space $C([0,\infty);X_{\mathrm{w}})$. A set $P \subset C([0,\infty);X_{\mathrm{w}})$ weakly attracts a set $Q \subset \mathcal{E}([0,\infty))$ if for any $\epsilon > 0$, there exists t_0, such that

$$T(t)Q \subset \{v \in C([0,\infty);X_{\mathrm{w}}) : \inf_{u\in P} \mathrm{d}_{C([0,\infty);X_{\mathrm{w}})}(u,v) < \epsilon\}, \quad \forall\, t \geq t_0.$$

Definition 3.2. *A set $P \subset C([0,\infty);X_{\mathrm{w}})$ is a weak trajectory attracting set for an evolutionary system $\mathcal{E}$ if it weakly attracts $\mathcal{E}([0,\infty))$.*

We have the following fact.

Lemma 3.3. *Let $P \subset C([0,\infty);X_{\mathrm{w}})$ satisfying $T(s)P = P, \forall\, s \geq 0$. P is a weak trajectory attracting set if and only if it satisfies the weak uniform tracking property.*

Proof. If P is a weak trajectory attracting set, then for any $\epsilon > 0$, there exists t_0, such that

$$T(t)\mathcal{E}([0,\infty)) \subset \{v \in C([0,\infty);X_{\mathrm{w}}) : \mathrm{d}_{C([0,\infty);X_{\mathrm{w}})}(P,v) < \epsilon\}, \ \ \forall\, t \geq t_0.$$

Hence, for any $t^* \geq t_0$ and every trajectory $u \in \mathcal{E}([0,\infty))$, we know that

$$\mathrm{d}_{C([0,\infty);X_{\mathrm{w}})}(T(t^*)u, v) < \epsilon, \tag{3}$$

for some $v \in P$. Due to the assumption, there is $v^* \in P$ satisfying $T(t^*)v^* = v$. It follows from (3) that,

$$\mathrm{d}_{C([0,\infty);X_{\mathrm{w}})}(T(t^*)u, T(t^*)v^*) < \epsilon. \tag{4}$$

By (2) of the definition of the translation operators $\{T(s)\}_{s\geq 0}$, (4) is equivalent to

$$\mathrm{d}_{C([t^*,\infty);X_{\mathrm{w}})}(u,v^*) < \epsilon. \tag{5}$$

Therefore, P satisfies the weak uniform tracking property.

Contrarily, suppose that for any $\epsilon > 0$, $u \in \mathcal{E}([0,\infty))$ and $t^* \geq t_0$ for some $t_0 \geq 0$, (5) holds for a $v^* \in P$. Hence, (4) is valid due to (2) again. Note that the assumption on P implies $T(t^*)v^* \in P$. Thus, (4) means that P is a weak trajectory attracting set. □

Definition 3.4. *A set $\mathfrak{A}_{\mathrm{w}} \subset C([0,\infty);X_{\mathrm{w}})$ is a weak trajectory attractor for an evolutionary system $\mathcal{E}$ if $\mathfrak{A}_{\mathrm{w}}$ is a minimal weak trajectory attracting set that is*

(1) *closed in $C([0,\infty);X_{\mathrm{w}})$.*
(2) *invariant: $T(t)\mathfrak{A}_{\mathrm{w}} = \mathfrak{A}_{\mathrm{w}}$, $\forall\, t \geq 0$.*

It is said that $\mathfrak{A}_{\mathrm{w}}$ is weakly compact if it is compact in $C([0,\infty);X_{\mathrm{w}})$.

It is easy to see that if a weak trajectory attractor exists, it is unique, and, by Lemma 3.3, if it is weakly compact, it is the minimal set that is closed in $C([0,\infty);X_{\mathrm{w}})$, invariant and satisfying the finite weak uniform tracking property.

The above definitions and results have versions of strong metric.

Definition 3.5. *A set $P \subset C([0,\infty);X_{\mathrm{w}})$ satisfies the strong uniform tracking property if for any $\epsilon > 0$ and $T > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*

$$\mathrm{d}_{\mathrm{s}}(u(t),v(t)) < \epsilon, \quad \forall t \in [t^*, t^*+T], \tag{6}$$

for some trajectory $v \in P$. Furthermore, if there exists a finite subset $P^f \subset P$, such that (6) holds for some $v \in P^f$, then P satisfies the finite strong uniform tracking property.

A set $P \subset C([0,\infty);X_{\mathrm{w}})$ strongly attracts a set $Q \subset \mathcal{E}([0,\infty))$ if for any $\epsilon > 0$ and $T > 0$, there exists t_0, such that

$$T(t)Q \subset \{v \in C([0,\infty);X_{\mathrm{w}}) : \inf_{u\in P} \mathrm{d}_{C([0,T];X_{\mathrm{s}})}(u,v) < \epsilon\}, \quad \forall\, t \geq t_0.$$

Definition 3.6. *A set $P \subset C([0,\infty);X_{\mathrm{w}})$ is a strong trajectory attracting set for an evolutionary system $\mathcal{E}$ if it strongly attracts $\mathcal{E}([0,\infty))$.*

Lemma 3.7. *A strong trajectory attracting set is a weak trajectory attracting set.*

Proof. Let P is a strong trajectory attracting set. Suppose that it is not a weak trajectory attracting set. Then, there exist $\epsilon_0 > 0$, and sequences $t_n \to \infty$ as $n \to \infty$, $u_n \in \mathcal{E}([0,\infty))$, such that

$$\mathrm{d}_{C([0,\infty);X_{\mathrm{w}})}(P, T(t_n)u_n) > 3\epsilon_0. \tag{7}$$

Let $l_0 \in \mathbb{N}$ satisfying

$$\sum_{l>l_0} \frac{1}{2^l} \le \epsilon_0.$$

By the definition of the metric of $\mathrm{d}_{C([0,\infty);X_{\mathrm{w}})}$, we obtain from (7) that

$$\mathrm{d}_{C([0,l_0];X_{\mathrm{w}})}(P, T(t_n)u_n) \sum_{l\le l_0} \frac{1}{2^l} + \sum_{l>l_0} \frac{1}{2^l} > 3\epsilon_0, \quad \forall\, n \in \mathbb{N},$$

which yields that

$$\mathrm{d}_{C([0,l_0];X_{\mathrm{w}})}(P, T(t_n)u_n) > 2\epsilon_0, \quad \forall\, n \in \mathbb{N}. \tag{8}$$

On the other hand, since P is a strong trajectory attracting set, we have that

$$\lim_{n\to\infty} \mathrm{d}_{C([0,l_0];X_{\mathrm{s}})}(P, T(t_n)u_n) = 0.$$

Passing to a subsequence and dropping a subindex, we can assume that there exists a sequence $v_n \in P$, such that

$$\lim_{n\to\infty} \mathrm{d}_{C([0,l_0];X_{\mathrm{s}})}(v_n, T(t_n)u_n) = 0. \tag{9}$$

Thanks to (8), there exists a sequence $\{s_n\} \subset [0, l_0]$, such that

$$\mathrm{d}_{\mathrm{w}}(v_n(s_n), (T(t_n)u_n)(s_n)) > \epsilon_0, \quad \forall\, n \in \mathbb{N}. \tag{10}$$

However, it follows from (9) that

$$\lim_{n\to\infty} \mathrm{d}_{\mathrm{s}}(v_n(s_n), (T(t_n)u_n)(s_n)) = 0,$$

which implies that

$$\lim_{n\to\infty} \mathrm{d}_{\mathrm{w}}(v_n(s_n), (T(t_n)u_n)(s_n)) = 0.$$

This contradicts to (10). We complete the proof. □

Similarly, we have the following.

Lemma 3.8. *Let $P \subset C([0,\infty); X_{\mathrm{w}})$ satisfying $T(s)P = P$, $\forall\, s \ge 0$. P is a strong trajectory attracting set if and only if it satisfies the strong uniform tracking property.*

Proof. The proof is similar to that of Lemma 3.3. We omit it. □

Definition 3.9. *A set $\mathfrak{A}_{\mathrm{s}} \subset C([0,\infty); X_{\mathrm{w}})$ is a strong trajectory attractor for an evolutionary system $\mathcal{E}$ if $\mathfrak{A}_{\mathrm{s}}$ is a minimal strong trajectory attracting set that is*

(1) *closed in $C([0,\infty); X_{\mathrm{w}})$.*
(2) *invariant: $T(t)\mathfrak{A}_{\mathrm{s}} = \mathfrak{A}_{\mathrm{s}}$, $\forall\, t \ge 0$.*

It is said that $\mathfrak{A}_{\mathrm{s}}$ is strongly compact if it is compact in $C([0,\infty); X_{\mathrm{s}})$.

By Lemma 3.7, if a strong trajectory attractor exists, it is a weak trajectory attractor. Hence, it is unique. Analogously, due to Lemma 3.8, a strongly compact strong trajectory attractor is the minimal set that is closed in $C([0,\infty);X_{\mathrm{w}})$, invariant and satisfying the finite strong uniform tracking property.

4. Attractor for evolutionary system

We will investigate evolutionary systems $\mathcal{E}$ satisfying the following property:

A1 $\mathcal{E}([0,\infty))$ is a precompact set in $C([0,\infty);X_{\mathrm{w}})$.

Such kinds of evolutionary systems are closely related to the concept of the uniform w.r.t. the initial time global attractor for a nonautonomous dynamical system, initiated by Haraux [Ha91]. For instance, the uniform global attractor for the evolutionary system defined by a nonautonomous PDE of mathematical physics, with Σ in Definition 2.5 taken as the translation family of the time-dependent items of the original equation, is the uniform w.r.t. the initial time global attractor for the same equation due to Haraux, and the evolutionary system satisfies A1 in general. For more details see [CL09, CL14].

Let

$$\bar{\mathcal{E}}([\tau,\infty)) := \overline{\mathcal{E}([\tau,\infty))}^{C([\tau,\infty);X_{\mathrm{w}})}, \quad \forall\, \tau\in\mathbb{R}.$$

It can be checked that $\bar{\mathcal{E}}$ is also an evolutionary system. We call $\bar{\mathcal{E}}$ the closure of the evolutionary system $\mathcal{E}$, and add the top-script $^-$ to the corresponding notations in previous sections for $\bar{\mathcal{E}}$. For example, we denote by $\bar{\mathcal{A}}_\bullet$ the uniform $\mathrm{d}_\bullet$-global attractor for $\bar{\mathcal{E}}$. Obviously, $\bar{\mathcal{E}}$ satisfies the following stronger version of A1:

$\bar{\mathrm{A}}1$ $\mathcal{E}([0,\infty))$ is a compact set in $C([0,\infty);X_{\mathrm{w}})$.

Note that for some evolutionary systems $\mathcal{E}$, say, those generated by autonomous dynamical systems, $\bar{\mathcal{E}}=\mathcal{E}$. However, instead of condition $\bar{\mathrm{A}}1$, the nonautonomous evolutionary systems $\mathcal{E}$ usually only satisfy A1. Moreover, for some nonautonomous evolutionary systems, as we will see next sections, there may not exist symbol spaces associated to their closures $\bar{\mathcal{E}}$. Since there is no necessity of constructing a symbol space, it is possible to investigate the properties of the global attractor for any $\mathcal{E}$ via those of the global attractor for its $\bar{\mathcal{E}}$, no matter whether it is nonuniqueness or lack of a symbol space.

With the observations in previous section in hand, Theorems 3.5 and 4.3 in [CL14] are retold in following form.

Theorem 4.1. *Let $\mathcal{E}$ be an evolutionary system. Then*

1. *The weak global attractor $\mathcal{A}_{\mathrm{w}}$ exists, and $\mathcal{A}_{\mathrm{w}}=\omega_{\mathrm{w}}(X)$.*

Furthermore, assume that $\mathcal{E}$ satisfies A1. Let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then

2. $\mathcal{A}_{\mathrm{w}}=\omega_{\mathrm{w}}(X)=\bar{\omega}_{\mathrm{w}}(X)=\bar{\omega}_{\mathrm{s}}(X)=\bar{\mathcal{A}}_{\mathrm{w}}$.

3. $\mathcal{A}_{\mathrm{w}}$ *is the maximal invariant and maximal quasi-invariant set w.r.t.* $\bar{\mathcal{E}}$*:*
$$\mathcal{A}_{\mathrm{w}} = \{u_0 \in X : u_0 = u(0) \text{ for some } u \in \bar{\mathcal{E}}((-\infty,\infty))\}.$$
4. *The weak trajectory attractor* $\mathfrak{A}_{\mathrm{w}}$ *exists, it is weakly compact, and*
$$\mathfrak{A}_{\mathrm{w}} = \Pi_+\bar{\mathcal{E}}((-\infty,\infty)) := \{u(\cdot)|_{[0,\infty)} : u \in \bar{\mathcal{E}}((-\infty,\infty))\}.$$
Hence, the finite weak uniform tracking property holds.
5. $\mathcal{A}_{\mathrm{w}}$ *is a section of* $\mathfrak{A}_{\mathrm{w}}$*:*
$$\mathcal{A}_{\mathrm{w}} = \mathfrak{A}_{\mathrm{w}}(t) := \{u(t) : u \in \mathfrak{A}_{\mathrm{w}}\}, \quad \forall\, t \geq 0.$$

Proof. The conclusions 1-3 are results of Theorem 3.5 in [CL14] and the existence of $\mathfrak{A}_{\mathrm{w}}$ and conclusion 5 are just Theorem 4.3 in [CL14]. Since $\mathfrak{A}_{\mathrm{w}}$ is invariant by definition, it follows from Lemma 3.3 that $\mathfrak{A}_{\mathrm{w}}$ satisfies the weak uniform tracking property, which is equivalent to the rest result 4 of Theorem 3.5 in [CL14]. Thanks to the assumption A1, $\mathfrak{A}_{\mathrm{w}}$ is weakly compact. Hence, the finite weak uniform tracking property holds. □

Definition 4.2. *The evolutionary system* $\mathcal{E}$ *is asymptotically compact if for any* $t_k \to +\infty$ *and any* $x_k \in R(t_k)X$*, the sequence* $\{x_k\}$ *is relatively strongly compact.*

We have a stronger version of Theorems 3.6 and 4.4 in [CL14].

Theorem 4.3. *Let* $\mathcal{E}$ *be an asymptotically compact evolutionary system. Then*

1. *The strong global attractor* $\mathcal{A}_{\mathrm{s}}$ *exists, it is strongly compact, and* $\mathcal{A}_{\mathrm{s}} = \mathcal{A}_{\mathrm{w}}$.

Furthermore, assume that $\mathcal{E}$ *satisfies A1. Let* $\bar{\mathcal{E}}$ *be the closure of* $\mathcal{E}$*. Then*

2. *The strong trajectory attractor* $\mathfrak{A}_{\mathrm{s}}$ *exists and* $\mathfrak{A}_{\mathrm{s}} = \mathfrak{A}_{\mathrm{w}}$*, it is strongly compact. Hence, the finite strong uniform tracking property holds, i.e., for any* $\epsilon > 0$ *and* $T > 0$*, there exist* t_0 *and a finite subset* $P^f \subset \mathfrak{A}_{\mathrm{s}} = \Pi_+\bar{\mathcal{E}}((-\infty,\infty))$*, such that for any* $t^* > t_0$*, every trajectory* $u \in \mathcal{E}([0,\infty))$ *satisfies* $\mathrm{d}_{\mathrm{s}}(u(t), v(t)) < \epsilon$*,* $\forall t \in [t^*, t^*+T]$*, for some trajectory* $v \in P^f$*.*

Proof. Conclusion 1 is that of Theorem 3.6 in [CL14]. Due to Theorem 4.4 in [CL14] and Definition 3.6, the weak trajectory attractor $\mathfrak{A}_{\mathrm{w}}$ is a strong trajectory attracting set that is invariant and compact in $C([0,\infty); X_{\mathrm{w}})$. If $P \subset C([0,\infty); X_{\mathrm{w}})$ is any other strong trajectory attracting set being invariant and closed in $C([0,\infty); X_{\mathrm{w}})$, we know form Lemma 3.7 that P is also a weakly compact weak trajectory attracting set. Hence $\mathfrak{A}_{\mathrm{w}} \subset P$. This concludes that, according to Definition 3.9, $\mathfrak{A}_{\mathrm{w}}$ is indeed a strong trajectory attractor $\mathfrak{A}_{\mathrm{s}}$. Similar to Theorem 4.1, by Lemma 3.8, $\mathfrak{A}_{\mathrm{s}}$ satisfies the strong uniform tracking property, which equals to the second part of Theorem 3.6 in [CL14].

The remains is to demonstrate the compactness of $\mathfrak{A}_{\mathrm{s}}$ in $C([0,\infty); X_{\mathrm{s}})$.

First, we have $\mathfrak{A}_{\mathrm{s}} \subset C([0,\infty); X_{\mathrm{s}})$. In fact, thanks to Theorem 4.1, for every $u \in \mathfrak{A}_{\mathrm{s}}$,
$$\{u(t) : t \in [0,\infty)\} \subset \mathcal{A}_{\mathrm{s}},$$

is compact in X_{s}. Hence, any weakly convergent sequence $u(t_n)$ with limit $u(t_0)$ as $t_n \to \infty$ does strongly converge to $u(t_0)$, which implies $u \in C([0,\infty); X_{\mathrm{s}})$.

Note that $\mathfrak{A}_{\mathrm{s}}$ is compact in $C([0,\infty); X_{\mathrm{w}})$. Now take a sequence $\{u_n(t)\} \subset \mathfrak{A}_{\mathrm{s}}$ that converges to $u(t)$ in $C([0,\infty); X_{\mathrm{w}})$. We claim that the convergence is indeed in $C([0,\infty); X_{\mathrm{s}})$. Otherwise, there exist $\epsilon > 0$, $T > 0$, and sequences $\{n_j\}$, $n_j \to \infty$ as $j \to \infty$ and $\{t_{n_j}\} \subset [0,T]$, such that

$$\mathrm{d}_{\mathrm{s}}\left(u_{n_j}(t_{n_j}), u(t_{n_j})\right) > \epsilon, \quad \forall n_j. \tag{11}$$

The sequences

$$\left\{u_{n_j}(t_{n_j})\right\}, \left\{u(t_{n_j})\right\} \subset \mathcal{A}_{\mathrm{s}},$$

are relatively strongly compact due to the strong compactness of $\mathcal{A}_{\mathrm{s}}$. Passing to a subsequence and dropping a subindex, we may assume that $\left\{u_{n_j}(t_{n_j})\right\}$ and $\left\{u(t_{n_j})\right\}$ are strongly convergent with limits x and y, respectively. We have that

$$\mathrm{d}_{\mathrm{w}}(x,y) \le \mathrm{d}_{\mathrm{w}}(u_{n_j}(t_{n_j}), x) + \mathrm{d}_{\mathrm{w}}(u_{n_j}(t_{n_j}), u(t_{n_j})) + \mathrm{d}_{\mathrm{w}}(u(t_{n_j}), y), \quad \forall n_j. \tag{12}$$

By the assumption,

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \mathrm{d}_{\mathrm{w}}\left(u_{n_j}(t), u(t)\right) = 0.$$

Together with

$$\lim_{j\to\infty} \mathrm{d}_{\mathrm{w}}(u_{n_j}(t_{n_j}), x) = 0 \quad \text{and} \quad \lim_{j\to\infty} \mathrm{d}_{\mathrm{w}}(u(t_{n_j}), y) = 0,$$

it follows from (12) that $x = y$, which is a contradiction to (11). □

Corollary 4.4. *Let $\mathcal{E}$ be an asymptotically compact evolutionary system satisfying A1, and $\bar{\mathcal{E}}$ be its closure. Then the kernel $\bar{\mathcal{E}}((-\infty,\infty))$ of $\bar{\mathcal{E}}$ is equicontinuous on $\mathbb{R}$, i.e.,*

$$\mathrm{d}_{\mathrm{s}}\left(v(t_1), v(t_2)\right) \le \theta\left(|t_1 - t_2|\right), \quad \forall\, t_1, t_2 \in \mathbb{R},\ \forall v \in \bar{\mathcal{E}}((-\infty,\infty)),$$

where $\theta(l)$ is a positive function tending to 0 as $l \to 0^+$.

Proof. First, by Theorem 4.3, $\Pi_+\bar{\mathcal{E}}((-\infty,\infty))$ is compact in $C([0,\infty); X_{\mathrm{s}})$.

Now, without loss of generality, we assume that $|t_1 - t_2| \le 1$. Hence, t_1, t_2 belong to some interval $[T, T+2]$. Denote by

$$\Pi_{[a,b]}\bar{\mathcal{E}}((-\infty,\infty)) := \{u(\cdot)|_{[a,b]} : u \in \bar{\mathcal{E}}((-\infty,\infty))\}.$$

Notice that

$$\{v(\cdot + T) : v(\cdot) \in \Pi_{[T,T+2]}\bar{\mathcal{E}}((-\infty,\infty))\} = \Pi_{[0,2]}\bar{\mathcal{E}}((-\infty,\infty)).$$

Thus, we need only to verify that $\Pi_{[0,2]}\bar{\mathcal{E}}((-\infty,\infty))$ is equicontinuous on $[0,2]$. Thanks to the Arzelá-Ascoli compactness criterion, this follows from the fact that $\Pi_{[0,2]}\bar{\mathcal{E}}((-\infty,\infty))$ is compact in $C([0,2]; X_{\mathrm{s}})$. □

Remark 4.5. *That we present Theorems 4.1 and 4.3 in such forms suggests the following comments.*

1. *Theorem 4.3 and Corollary 4.4 indicate that the notion of a strongly compact strong trajectory attractor is an apt description of the results of finite strong uniform tracking property and equicontinuity.*
2. *Comparing with Theorem 4.1, Theorem 4.3 implies that the strong compactness of strong global attractor and the strong compactness of strong trajectory attractor follow simultaneously once we obtain the asymptotical compactness!*
3. *Theorems 4.1 and 4.3 show that a global attractor is a section of a trajectory attractor; the notion of a global attractor emphasizes the property of attracting trajectories while the notion of a trajectory attractor emphasizes the uniform tracking property. Hence, it is convenient to call them both the weak/strong attractor for the evolutionary system.*

We apply to, for instance, the 2D Navier-Stokes equations with non-slip boundary condition. If the external force is normal (see Section 5) in $L^2_{\mathrm{loc}}(\mathbb{R}; V')$, where V' is the dual of the space of divergence-free vector fields with square-integrable derivatives and vanishing on the boundary, all Leray-Hopf weak solutions form an evolutionary system satisfying A1 and its asymptotical compactness was actually obtained in [Lu06]. Therefore, Theorem 4.3 now provides the existence of a strongly compact strong trajectory attractor. As a consequence, it satisfies the finite strong uniform tracking property and all bounded complete Leray-Hopf weak solutions is equicontinuity. In 3D case, once the strong continuity of the trajectories in the weak trajectory attractor has been proved (see [CL14]), the similar results follow.

Now, we recall a method to verify the asymptotical compactness of evolutionary systems satisfying these additional properties:

A2 (Energy inequality) Assume that X is a set in some Banach space H satisfying the Radon-Riesz property (see below) with the norm denoted by $|\cdot|$, such that $\mathrm{d_s}(x,y) = |x-y|$ for $x, y \in X$ and $\mathrm{d_w}$ induces the weak topology on X. Assume also that for any $\epsilon > 0$, there exists $\delta > 0$, such that for every $u \in \mathcal{E}([0,\infty))$ and $t > 0$,

$$|u(t)| \le |u(t_0)| + \epsilon,$$

for t_0 a.e. in $(t-\delta, t)$.

A3 (Strong convergence a.e.) Let $u_k \in \mathcal{E}([0,\infty))$ be such that, u_k is $\mathrm{d}_{C([0,T];X_{\mathrm{w}})}$-Cauchy sequence in $C([0,T]; X_{\mathrm{w}})$ for some $T > 0$. Then $u_k(t)$ is $\mathrm{d_s}$-Cauchy sequence a.e. in $[0,T]$.

A Banach space $\mathcal{B}$ satisfies the Radon-Riesz property if

$$x_n \to x \text{ strongly in } \mathcal{B} \Leftrightarrow \begin{cases} x_n \to x \text{ weakly in } \mathcal{B} \\ \|x_n\|_{\mathcal{B}} \to \|x\|_{\mathcal{B}} \end{cases}.$$

In many applications, H in A2 is a uniformly convex separable Banach space and X is a bounded closed set in H. Then the weak topology of H is metrizable on X, and X is compact w.r.t. such a metric $\mathrm{d_w}$. Moreover, the Radon-Riesz property is automatically satisfied in this case.

Theorem 4.6. [CL14] *Let $\mathcal{E}$ be an evolutionary system satisfying A1, A2, and A3, and assume that its closure $\bar{\mathcal{E}}$ satisfies $\bar{\mathcal{E}}((-\infty,\infty)) \subset C((-\infty,\infty); X_{\mathrm{s}})$. Then $\mathcal{E}$ is asymptotically compact.*

5. Attractors for Reaction-diffusion systems

We study the long-time behavior of the solutions of the following nonautonomous reaction-diffusion system (RDS):

$$\begin{aligned} \partial_t u - a\Delta u + f(u,t) &= g(x,t), && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \\ u|_{t=\tau} &= u_\tau, && \tau \in \mathbb{R}. \end{aligned} \tag{RDS}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of sufficient smoothness; $a = \{a_{ij}\}_{i=1,\cdots,N}^{j=1,\cdots,N}$ is an $N\times N$ real matrix with positive symmetric part $\frac{1}{2}(a+a^*) \geq \beta I$, $\beta > 0$; $u = u(x,t) = (u^1,\cdots,u^N)$ is the unknown function, $g = (g^1,\cdots,g^N)$ is the driving force and $f = (f^1,\cdots,f^N)$ is the interaction function. Denote the spaces $H = \left(L^2(\Omega)\right)^N$ and $V = \left(H_0^1(\Omega)\right)^N$ with norms $|\cdot|$ and $\|\cdot\|$, respectively. Let V' be the dual of V. Assume that $g(s) = g(\cdot,s)$ is translation bounded (tr.b.) in $L^2_{\mathrm{loc}}(\mathbb{R};V')$, i.e.,

$$\|g\|^2_{L^2_{\mathrm{b}}} = \|g\|^2_{L^2_{\mathrm{b}}(\mathbb{R};V')} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g(s)\|^2_{V'}\, ds < \infty, \tag{13}$$

and $f \in C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ satisfies the following conditions:

$$\sum_{i=1}^N \gamma_i |v^i|^{p_i} - C \leq \sum_{i=1}^N f^i(v,s)v^i = f(v,s)\cdot v,\ \gamma_k > 0,\ \forall v \in \mathbb{R}^N, \tag{14}$$

$$\sum_{i=1}^N |f^i(v,s)|^{\frac{p_i}{p_i-1}} \leq C\left(\sum_{i=1}^N |v^i|^{p_i} + 1\right),\ \forall v \in \mathbb{R}^N, \tag{15}$$

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq 2,$$

where the letter C denotes a constant which may be different in each occasion throughout this paper.[1]

[1](RDS) with other boundary conditions such as Neumann or periodic boundary conditions can be handled in the same way, and all results hold for these boundary conditions. For the Dirichlet boundary conditions, instead of considering $p_k \geq 2$, $k = 1,\cdots,N$, for simplicity, we may assume that $p_k > 1$. See Remark II.4.1 and II.4.2 in [CV02] for more details.

Let $q_k := p_k/(p_k - 1)$, $r_k := \max\{1, n(1/2 - 1/p_k)\}$, $k = 1, \cdots, N$, and denote $\mathbf{p} := (p_1, \cdots, p_N)$, $\mathbf{q} := (q_1, \cdots, q_N)$, $\mathbf{r} := (r_1, \cdots, r_N)$ and

$$
\begin{aligned}
&L^{\mathbf{p}}(\Omega) := L^{p_1}(\Omega) \times L^{p_2}(\Omega) \times \cdots \times L^{p_N}(\Omega),\\
&H^{-\mathbf{r}}(\Omega) := H^{-r_1}(\Omega) \times H^{-r_2}(\Omega) \times \cdots \times H^{-r_N}(\Omega),\\
&L^{\mathbf{p}}(\tau, T; L^{\mathbf{p}}(\Omega)) := L^{p_1}(\tau, T; L^{p_1}(\Omega)) \times \cdots \times L^{p_N}(\tau, T; L^{p_N}(\Omega)),\\
&L^{\mathbf{q}}(\tau, T; H^{-\mathbf{r}}(\Omega)) := L^{q_1}(\tau, T; H^{-r_1}(\Omega)) \times \cdots \times L^{q_N}(\tau, T; H^{-r_N}(\Omega)).
\end{aligned}
$$

We recall the results on the existence of weak solutions to (RDS) (see e.g. [CV02]). Note that conditions (14)-(15) do not assure the uniqueness of the solutions.

Theorem 5.1. *For every $u_\tau \in H$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, there exists a weak solution $u(t)$ of (RDS) satisfying*

$$u \in C([\tau, \infty); H) \cap L^2_{\mathrm{loc}}(\tau, \infty; V) \cap L^{\mathbf{p}}_{\mathrm{loc}}(\tau, \infty; L^{\mathbf{p}}(\Omega)). \tag{16}$$

Moreover, the function $|u(t)|^2$ is absolutely continuous on $[\tau, \infty)$ and

$$\frac{d}{dt}|u(t)|^2 + (a\nabla u(t), \nabla u(t)) + (f(u(t), t), u(t)) = \langle g(t), u(t)\rangle, \tag{17}$$

for a.e. $t \in [\tau, \infty)$.

Now, we consider a fixed interaction function f_0 and a fixed driving force g_0, such that $f_0(v, t)$ satisfies (14)-(15) , and $g_0(t) \in L^2_{\mathrm{b}}(\mathbb{R}; V')$. Let $\sigma_0 := (f_0, g_0)$ and $\Sigma := \{\sigma_0(\cdot + h) : h \in \mathbb{R})\}$. Thus, for every $\sigma = (f, g) \in \Sigma$, f satisfies (14)-(15) with the same constants, and

$$\|g\|^2_{L^2_{\mathrm{b}}} \le \|g_0\|^2_{L^2_{\mathrm{b}}}.$$

Let $u(t)$, $t \in [\tau, \infty)$, be a weak solution of (RDS) with $\sigma = (f, g) \in \Sigma$ guaranteed by Theorem 5.1. Thanks to (14), we obtain from (17) that

$$\frac{d}{dt}|u(t)|^2 + \lambda_1\beta|u(t)|^2 \le C + \beta^{-1}\|g_0\|^2_{V'}, \tag{18}$$

for a.e. $t \in [\tau, \infty)$. Here λ_1 is the first eigenvalue of the Laplacian with Dirichlet boundary conditions. Due to the absolute continuity of $|u(t)|$ and Gronwall's inequality, (18) implies that

$$|u(t)|^2 \le |u(\tau)|^2 e^{-\lambda_1\beta(t-\tau)} + C, \qquad \forall\, t \in [\tau, \infty).$$

Therefore there exists a closed (uniform w.r.t. $\tau \in \mathbb{R}$) absorbing ball $B_{\mathrm{s}}(0, R)$, where the radius R depends on λ_1, β, the constant in (14) and $\|g_0\|^2_{L^2_{\mathrm{b}}}$. We denote by X the absorbing ball

$$X = \{u \in H : |u| \le R\}. \tag{19}$$

That is, for any bounded set $A \subset H$, there exists a time $t_0 \ge \tau$, such that

$$u(t) \in X, \qquad \forall\, t \ge t_0,$$

for every weak solution $u(t)$ with $\sigma \in \Sigma$ and the initial data $u(\tau) \in A$. It is known that X is weakly compact and metrizable with a metric $\mathrm{d_w}$ deducing the weak topology on X.

Consider an evolutionary system for which a family of trajectories consists of all weak solutions of (RDS) with the fixed σ_0 in X. More precisely, define

$$\begin{aligned}\mathcal{E}([\tau,\infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } [\tau,\infty) \\ \text{with } \sigma \in \Sigma \text{ and } u(t) \in X, \forall\, t \in [\tau,\infty)\}, \quad \tau \in \mathbb{R},\\ \mathcal{E}((-\infty,\infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } (-\infty,\infty) \\ \text{with } \sigma \in \Sigma \text{ and } u(t) \in X, \forall\, t \in (-\infty,\infty)\}.\end{aligned}$$

Clearly, the properties 1–4 in Definition 2.1 hold for $\mathcal{E}$ if we utilize the translation identity: the solutions of (RDS) with σ initiating at $\tau + h$ are also the solutions of (RDS) with $\sigma(\cdot + h)$ initiating at τ.

Thanks to Theorem 4.1, the weak global attractor $\mathcal{A}_\mathrm{w}$ for this evolutionary system exists.

Lemma 5.2. *Let $u_n(t)$ be a sequence of weak solutions of (RDS) with $\sigma_n \in \Sigma$, such that $u_n(t) \in X$ for all $t \geq t_0$. Then*

$$\begin{aligned} u_n \ \textit{ is bounded in } \ L^2(t_0,T;V),\\ \partial_t u_n \ \textit{ is bounded in } \ L^{\mathbf{q}}(t_0,T;H^{-\mathbf{r}}(\Omega)),\end{aligned}$$

for all $T > t_0$. Moreover, there exists a subsequence u_{n_j} converges in $C([t_0,T];H_\mathrm{w})$ to some $\phi(t) \in C([t_0,T];H)$, i.e.,

$$(u_{n_j}, v) \to (\phi, v) \ \textit{ uniformly on } \ [t_0,T],$$

as $n_j \to \infty$, for all $v \in H$.

Proof. Standard estimates (see e.g. [CV02]) show that

$$\{u_n\} \ \text{ is bounded in } \ L^2(t_0,T;V) \cap L^\infty(t_0,T;H) \cap L^{\mathbf{p}}\left(t_0,T;L^{\mathbf{p}}(\Omega)\right), \tag{20}$$

and

$$\{\partial_t u_n\} \ \text{ is bounded in } \ L^{\mathbf{q}}(t_0,T;H^{-\mathbf{r}}(\Omega)), \tag{21}$$

for all $T > t_0$. By the embedding theorem (cf. Theorem II.1.4 in [CV02], Theorem 8.1 in [Ro01]), we obtain that

$$\{u_n\} \ \text{ is precompact in } \ L^2(t_0,T;H). \tag{22}$$

Passing to a subsequence and dropping a subindex, we know from (20)-(22) that,

$$\begin{aligned} u_n(t) \to \phi(t) \quad & \text{weak-star in } L^\infty(t_0,T;H),\\ & \text{weakly in } L^2(t_0,T;V) \cap L^{\mathbf{p}}\left(t_0,T;L^{\mathbf{p}}(\Omega)\right),\\ & \text{strongly in } L^2(t_0,T;H),\end{aligned} \tag{23}$$

and

$$
\begin{aligned}
\Delta u_n(t) &\to \Delta\phi(t) \quad \text{weakly in } L^2(t_0,T;V'),\\
\partial_t u_n(t) &\to \partial_t\phi(t) \quad \text{weakly in } L^{\mathbf{q}}(t_0,T;H^{-\mathbf{r}}(\Omega)),\\
f_n(u_n(x,t),t) &\to \psi(t) \quad \text{weakly in } L^{\mathbf{q}}\left(t_0,T;L^{\mathbf{q}}(\Omega)\right),
\end{aligned}
\tag{24}
$$

for some

$$\phi(t)\in L^\infty(t_0,T;H)\cap L^2(t_0,T;V)\cap L^{\mathbf{p}}\left(t_0,T;L^{\mathbf{p}}(\Omega)\right),$$

and some

$$\psi(t)\in L^{\mathbf{q}}\left(t_0,T;L^{\mathbf{q}}(\Omega)\right).$$

Note that g_0 is translation compact in $L^{2,\mathrm{w}}_{\mathrm{loc}}(\mathbb{R};V')$, i.e. the translation family $\{g_0(\cdot+h):h\in\mathbb{R}\}$ is precompact in $L^{2,\mathrm{w}}_{\mathrm{loc}}(\mathbb{R};V')$ (see [CV02]). Thus, passing to a subsequence and dropping a subindex again, we also have,

$$g_n(t)\to g(t) \quad \text{weakly in } L^2(t_0,T;V') \tag{25}$$

with some $g(t)\in L^2(t_0,T;V')$. Passing the limits yields the following equality

$$\partial_t\phi - a\Delta\phi+\psi = g \tag{26}$$

in the distribution sense of the space $\mathcal{D}'(t_0,T;H^{-\mathbf{r}}(\Omega))$. Thanks to a vector version of Theorem II.1.8 in [CV02], (23)-(26) indicate that $\phi(t)\in C([t_0,T];H)$. On the other hand, by the convergence of (23), we know that $u_n(t)\to\phi(t)$ in $C([t_0,T];H_{\mathrm{w}})$. We complete the proof. $\square$

Now we give the definition of a normal function which was introduced in [LWZ05] and Lu [Lu06].

Definition 5.3. *Let $\mathcal{B}$ be a Banach space. A function $\varphi(s)\in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{B})$ is said to be normal in $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{B})$ if for any $\epsilon>0$, there exists $\delta>0$, such that*

$$\sup_{t\in\mathbb{R}}\int_t^{t+\delta}\|\varphi(s)\|^2_{\mathcal{B}}\,ds\le\epsilon.$$

Then, we have the following.

Lemma 5.4. *The evolutionary system $\mathcal{E}$ of (RDS) with the fixed σ_0 satisfies A1 and A3. Moreover, if g_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R};V')$ then A2 holds.*

Proof. The proof is analogous to that of Lemma 3.4 in [CL09]. We write it out for completeness and for reader's convenience. First, by Theorem 5.1, $\mathcal{E}([0,\infty))\subset C([0,\infty);X_{\mathrm{s}})$. Now take any sequence $u_n\in\mathcal{E}([0,\infty))$, $n=1,2,\cdots$. Owing to Lemma 5.2, there exists a subsequence, still denoted by u_n, which converges in $C([0,1];X_{\mathrm{w}})$ to some $\phi^1\in C([0,1];X_{\mathrm{s}})$ as $n\to\infty$. Passing to a subsequence and dropping a subindex once more, we have that $u_n\to\phi^2$ in $C([0,2];X_{\mathrm{w}})$ as $n\to\infty$ for some $\phi^2\in C([0,2];X_{\mathrm{s}})$. Note that $\phi^1(t)=\phi^2(t)$ on $[0,1]$. Continuing

this diagonalization process, we obtain a subsequence u_{n_j} of u_n that converges in $C([0,\infty);X_{\mathrm{w}})$ to some $\phi \in C([0,\infty);X_{\mathrm{s}})$ as $n_j \to \infty$. Therefore, A1 holds.

Let $u_k \in \mathcal{E}([0,\infty))$ be such that u_k is a $\mathrm{d}_{C([0,T];X_{\mathrm{w}})}$-Cauchy sequence in $C([0,T];X_{\mathrm{w}})$ for some $T > 0$. Thanks to Lemma 5.2 again, the sequence $\{u_k\}$ is bounded in $L^2(0,T;V)$. Hence, there exists some $\phi(t) \in C([0,T];X_{\mathrm{w}})$, such that

$$\int_0^T |u_k(s) - \phi(s)|^2 \, ds \to 0, \quad \text{as} \quad k \to \infty.$$

In particular, $|u_k(t)| \to |\phi(t)|$ as $k \to \infty$ a.e. on $[0,T]$, which means that $u_k(\cdot)$ is d_{s}-Cauchy sequence a.e. on $[0,T]$. Thus, A3 is valid.

For any $u \in \mathcal{E}([0,\infty))$ and $t > 0$, it follows from (18) and the absolute continuity of $|u(t)|^2$ that

$$|u(t)|^2 \le |u(t_0)|^2 + C(t-t_0) + \frac{1}{\beta}\int_{t_0}^t \|g_0\|_{V'}^2 \, ds, \tag{27}$$

for all $0 \le t_0 < t$. Here C is independent of u. Suppose now that g_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R};V')$. Then given $\epsilon > 0$, there exists $0 < \delta < \frac{\epsilon}{2C}$, such that

$$\sup_{t\in\mathbb{R}} \int_{t-\delta}^t \|g_0(s)\|_{V'}^2 \, ds \le \frac{\beta\epsilon}{2}.$$

Hence, we obtain from (27) that

$$|u(t)|^2 \le |u(t_0)|^2 + \epsilon, \quad \forall t_0 \in (t-\delta, t),$$

which concludes that A2 holds. $\square$

Let $\bar{\mathcal{E}}$ be the closure of the evolutionary system $\mathcal{E}$, i.e.,

$$\bar{\mathcal{E}}([\tau,\infty)) := \overline{\mathcal{E}([\tau,\infty))}^{C([\tau,\infty);X_{\mathrm{w}})}, \quad \forall\, \tau \in \mathbb{R}.$$

Then, it follows from Theorem 5.1, Lemma 5.4, Theorems 4.1, 4.6 and 4.3 that

Theorem 5.5. *The uniform weak global attractor $\mathcal{A}_{\mathrm{w}}$ and the weak trajectory attractor $\mathfrak{A}_{\mathrm{w}}$ for (RDS) with a fixed interaction function f_0 and a fixed driving force g_0 satisfying (13)-(15) exist, and*

$$\begin{aligned}
&\mathcal{A}_{\mathrm{w}} = \omega_{\mathrm{w}}(X) = \omega_{\mathrm{s}}(X) = \{u(0) : u \in \bar{\mathcal{E}}((-\infty,\infty))\},\\
&\mathfrak{A}_{\mathrm{w}} = \Pi_+\bar{\mathcal{E}}((-\infty,\infty)) = \{u(\cdot)|_{[0,\infty)} : u \in \bar{\mathcal{E}}((-\infty,\infty))\},\\
&\mathcal{A}_{\mathrm{w}} = \mathfrak{A}_{\mathrm{w}}(t) = \{u(t) : u \in \mathfrak{A}_{\mathrm{w}}\}, \quad \forall\, t \ge 0.
\end{aligned}$$

Moreover, $\mathfrak{A}_{\mathrm{w}}$ satisfies the finite weak uniform tracking property and is weakly equicontinuity on $[0,\infty)$.

Theorem 5.6. *Furthermore, if g_0 is normal in $L^2_{\mathrm{loc}}(\mathbb{R};V')$, then the uniform weak global attractor $\mathcal{A}_{\mathrm{w}}$ is a strongly compact strong global attractor $\mathcal{A}_{\mathrm{s}}$, and the weak trajectory attractor $\mathfrak{A}_{\mathrm{w}}$ is a strongly compact strong trajectory attractor $\mathfrak{A}_{\mathrm{s}}$. Hence,*

$\mathfrak{A}_{\mathrm{s}}$ *satisfies the finite strong uniform tracking property and is strongly equicontinuity on* $[0,\infty)$.

6. On nonlinearity

In this section, for the reader's convenience, we begin with a brief review of some additional assumptions, other than (14) and (15), on the nonlinearity to obtain the existence and the structure of uniform global attractors or trajectory attractors for (RDS) in some previous literature (see [CV94, CV96, CV97, CV02, Lu07, CL09]. Then, we examine the results on attractors using our framework and discuss their relations to those obtained in previous section. Finally, we construct some examples that do not satisfy these extra restrictions.

Let $\mathcal{M}_1$ be $C(\mathbb{R}^N;\mathbb{R}^N)$ endowed with finite weighted norm

$$\|\phi\|_{\mathcal{M}_1} = \sup_{v\in\mathbb{R}^N}\left(\sum_{i=1}^N \frac{|\phi^i(v)|}{\left(1+\Sigma_{j=1}^N|v^j|^{p_j}\right)^{\frac{p_i-1}{p_i}}}\right).$$

Chepyzhov and Vishik studied the uniform global attractor of (RDS) in [CV94] (see also [CV02]) assuming that $f_0(v,s)$ is translation compact in $C(\mathbb{R};\mathcal{M}_1)$, that is, the closure of the translation family $\{f_0(\cdot,\cdot+h) : h\in\mathbb{R}\}$ in $C(\mathbb{R};\mathcal{M}_1)$ is compact in $C(\mathbb{R};\mathcal{M}_1)$. Later on, they investigated (RDS) in [CV96, CV97] (see also [CV02]) by the method of the so-called trajectory attractor. The condition on $f_0(v,s)$ is translation compactness in $C(\mathbb{R};\mathcal{M})$, where $\mathcal{M} = C(\mathbb{R}^N;\mathbb{R}^N)$ is endowed with the topology of local uniform convergence. This restriction on f_0 is equivalent to that $f_0(v,s)$ is bounded and uniformly continuous in every cylinder $Q(R)=\{(v,s) : \|v\|_{\mathbb{R}^N}\le R, s\in\mathbb{R}\}$, $R>0$. The section of the (weak) trajectory attractor at time $t=0$ is the weak uniform w.r.t. the symbol space $\bar{\Sigma}$ global attractor. Here one component of $\bar{\Sigma}$ is the closure of $\{f_0(\cdot,\cdot+h) : h\in\mathbb{R}\}$ in $C(\mathbb{R};\mathcal{M})$, which is compact in $C(\mathbb{R};\mathcal{M})$. However, as we will see below, its relation to the uniform global attractor for the originally considered (RDS) with a fixed interaction function f_0 and a fixed driving force g_0 is not yet clear.

In [Lu07], the author obtained the existence and the structure of the uniform global attractor of (RDS) with a weaker condition (see Theorems 6.1 and 6.2 below) on the nonlinearity.

Let $C^{\mathrm{p.u.}}(\mathbb{R};\mathcal{M})$ denote the space $C(\mathbb{R};\mathcal{M})$ endowed with the topology of following convergence: $\varphi_n(s)\to\varphi(s)$ as $n\to\infty$ in $C^{\mathrm{p.u.}}(\mathbb{R};\mathcal{M})$, if $\varphi_n(v,s)$ is uniformly bounded on any ball in $\mathbb{R}^N\times\mathbb{R}$ and for every $s\in\mathbb{R}$, $R>0$,

$$\max_{\|v\|_{\mathbb{R}^N}\le R}\|\varphi_n(v,s)-\varphi(v,s)\|_{\mathbb{R}^N}\to 0,\quad \text{as } n\to\infty.$$

Denote by $C^{\mathrm{p.p.}}(\mathbb{R};\mathcal{M})$ the space $C(\mathbb{R};\mathcal{M})$ endowed with another topology of following convergence: $\varphi_n(s)\to\varphi(s)$ as $n\to\infty$ in $C^{\mathrm{p.p.}}(\mathbb{R};\mathcal{M})$, if $\varphi_n(v,s)$ is

uniformly bounded on any ball in $\mathbb{R}^N \times \mathbb{R}$ and for every $(v,s) \in \mathbb{R}^N \times \mathbb{R}$,

$$\|\varphi_n(v,s) - \varphi(v,s)\|_{\mathbb{R}^N} \to 0, \quad \text{as } n \to \infty.$$

Note that $C^{\text{p.p.}}(\mathbb{R};\mathcal{M})$ is in fact $C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ endowed with the usual weak topology.

Let $C^{\text{p.u.}}_{\text{tr.c.}}(\mathbb{R};\mathcal{M})$, $C^{\text{p.p.}}_{\text{tr.c.}}(\mathbb{R};\mathcal{M})$ and $C_{\text{tr.c.}}(\mathbb{R};\mathcal{M})$ be the classes of the translation compact functions in $C^{\text{p.u.}}(\mathbb{R};\mathcal{M})$, $C^{\text{p.p.}}(\mathbb{R};\mathcal{M})$ and $C(\mathbb{R};\mathcal{M})$, respectively.

The functions in $C^{\text{p.u.}}_{\text{tr.c.}}(\mathbb{R};\mathcal{M})$ are characterized by the following.

Theorem 6.1. [Lu07] $\varphi(s) \in C^{\text{p.u.}}_{\text{tr.c.}}(\mathbb{R};\mathcal{M})$ *if and only if* $\varphi(s) \in C^{\text{p.p.}}_{\text{tr.c.}}(\mathbb{R};\mathcal{M})$ *and one of the following holds.*
(i) $\{\varphi(s)|s \in \mathbb{R}\}$ *is precompact in* $\mathcal{M}$*.*
(ii) For any $R > 0$*,* $\varphi(v,s)$ *is bounded in* $Q(R) = \{(v,s) : \|v\|_{\mathbb{R}^N} \le R, s \in \mathbb{R}\}$*, and*

$$\|\varphi(v_1,s) - \varphi(v_2,s)\|_{\mathbb{R}^N} \le \theta\left(\|v_1 - v_2\|_{\mathbb{R}^N}, R\right),\ \forall\,(v_1,s),(v_2,s) \in Q(R), \tag{28}$$

where $\theta(l,R)$ *is a positive function tending to* 0 *as* $l \to 0^+$*.*

By Arzelá-Ascocli compactness criterion, the conditions of (i) and (ii) imply that the family $\{\varphi(\cdot,s) : s \in \mathbb{R}\}$ is equicontinuous on any ball $\{v : \|v\|_{\mathbb{R}^N} \le R\}$.

Let $C_{\text{b}}(\mathbb{R};\mathcal{M})$ be the space of bounded continuous functions with values in $\mathcal{M}$ and endowed with the uniform convergence topology on $\mathbb{R}$. We have the following relationships.

Theorem 6.2. [Lu07] $C_{\text{tr.c.}}(\mathbb{R};\mathcal{M}) \subset C^{\text{p.u.}}_{\text{tr.c.}}(\mathbb{R};\mathcal{M}) \subset C^{\text{p.p.}}_{\text{tr.c.}}(\mathbb{R};\mathcal{M}) \subset C_{\text{b}}(\mathbb{R};\mathcal{M})$ *with all inclusions being proper and the former three sets being closed in* $C_{\text{b}}(\mathbb{R};\mathcal{M})$*.*

In [CL09], Cheskidov and Lu generalized the results in [Lu07] to (RDS) without uniqueness and considered in addition the weak uniform global attractors. More precisely, besides the conditions on f_0 and g_0 in Section 5, we suppose more that $f_0 \in C^{\text{p.u.}}_{\text{tr.c.}}(\mathbb{R};\mathcal{M})$. Denote again by $\sigma_0 = (f_0, g_0)$. The family of (RDS) with $\sigma = (f,g)$ belonging to the following symbol space

$$\bar{\Sigma} = \overline{\{(\sigma_0(\cdot + h) : h \in \mathbb{R})\}}^{C^{\text{p.u.}}(\mathbb{R};\mathcal{M}) \times L^{2,\text{w}}_{\text{loc}}(\mathbb{R};V')}$$

defines an evolutionary system $\mathcal{E}_{\bar{\Sigma}}$ satisfying $\bar{\text{A}}1$. The existence and the structure of its uniform global attractor $\mathcal{A}^{\bar{\Sigma}}_{\bullet}$ is presented in [CL09]. Now we apply Theorems 4.1 and 4.3 to $\mathcal{E}_{\bar{\Sigma}}$, we can get more the trajectory attractor $\mathfrak{A}^{\bar{\Sigma}}_{\bullet}$. Similarly defined as in Section 5, we have the evolutionary system $\mathcal{E}$ and its closure $\bar{\mathcal{E}}$ for the originally considered (RDS) with the fixed σ_0. Evidently, $\mathcal{E} \subset \bar{\mathcal{E}} \subset \mathcal{E}_{\bar{\Sigma}}$. Hence,

$$\mathcal{A}_{\bullet} = \bar{\mathcal{A}}_{\bullet} \subset \mathcal{A}^{\bar{\Sigma}}_{\bullet} \quad \text{or} \quad \mathfrak{A}_{\bullet} = \bar{\mathfrak{A}}_{\bullet} \subset \mathfrak{A}^{\bar{\Sigma}}_{\bullet}.$$

However, it is not known whether the following identities hold

$$\mathcal{A}_{\bullet} = \bar{\mathcal{A}}_{\bullet} = \mathcal{A}^{\bar{\Sigma}}_{\bullet}\ ? \tag{29}$$

Or in the version of trajectory attractors,

$$\mathfrak{A}_{\bullet} = \bar{\mathfrak{A}}_{\bullet} = \mathfrak{A}_{\bullet}^{\bar{\Sigma}} \ ?$$

In [CL09] (see also [Lu07]), it is shown that (29) is valid, if we suppose further the following condition on nonlinearity f_0,

$$(f_0(v_1, s) - f_0(v_2, s), v_1 - v_2) \geq -C\|v_1 - v_2\|_{\mathbb{R}^N}^2, \forall v_1, v_2 \in \mathbb{R}^N, \forall s \in \mathbb{R}, \tag{30}$$

which guarantees the uniqueness of the solutions. In fact, the general result for evolutionary systems with uniqueness is proved in [CL14]. Contrarily, $\mathcal{A}_{\bullet}^{\bar{\Sigma}}$ or $\mathfrak{A}_{\mathrm{w}}^{\bar{\Sigma}}$, previously constructed for the system without uniqueness, might not satisfy the minimality property w.r.t. uniformly (w.r.t. initial time) attracting for the original system. This means that they might be bigger than the uniform global attractor or the trajectory attractor followed by our framework.

Now we construct several examples in $C(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ that satisfy conditions (14)-(15) but not those in Theorem 6.1 and (30). It is not clear how to obtain the results in Theorems 5.5 and 5.6 for the (RDS) with such kind of nonlinearities by previous frameworks.

Let $T = \max\{0, t\}$, $t \in \mathbb{R}$.

Example I.

$$f(v,t) = \begin{cases} |v|^p, & \text{if } v \leq 0, \\ (1+T)v, & \text{if } 0 \leq v \leq \frac{1}{1+T}, \\ \left|v - \frac{1}{1+T}\right|^p + 1, & \text{if } v > \frac{1}{1+T}. \end{cases}$$

Note that, the family $\{f(\cdot, t) : t \in \mathbb{R}\}$ is not equicontinuous on $[0, 1]$, which means that $f(v, t)$ does not satisfy (28). Moreover, the pointwise limit function of $f(\cdot, t)$, as $t \to +\infty$, is a discontinuous function,

$$f_\infty(v) = \begin{cases} |v|^p, & \text{if } v \leq 0, \\ |v|^p + 1, & \text{if } v > 0. \end{cases}$$

Hence, $f(v, t)$ does not even belong to $C_{\text{tr.c.}}^{\text{p.p.}}(\mathbb{R}; \mathcal{M})$, where $\mathcal{M} = C(\mathbb{R}, \mathbb{R})$. In fact, for any sequence $\{f(\cdot, \cdot + t_n), t_n \to +\infty\}$, the pointwise limit is f_∞.

Example II.

$$f(v,t) = \begin{cases} |v + 2\pi|^p, & \text{if } v \leq -2\pi, \\ \rho(v)\sin(1+T)v, & \text{if } -2\pi < v < 2\pi, \\ |v - 2\pi|^p, & \text{if } v \geq 2\pi, \end{cases}$$

where $\rho(\cdot)$ is a continuous function supported on $[-2\pi, 2\pi]$. For instance, $\rho(\cdot)$ is an infinitely differentiable function supported on $(-2\pi, 2\pi)$ and equals to 1 on $[-\pi, \pi]$. Note again that, the family $\{f(\cdot, t) | t \in \mathbb{R}\}$ is not equicontinuous in $[-2\pi, 2\pi]$.

Moreover, there is no a pointwise limit function of any sequence $\{f(\cdot, \cdot + t_n)\}$, as $t_n \to +\infty$. Hence, $f(v,t) \notin C^{\text{p.p.}}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$.

Example III.

$$f\left(v,t\right) = \begin{cases} |v+2|^p, & \text{if } v \leq -2, \\ \rho(v)\sin T^2, & \text{if } -2 < v < 2, \\ |v-2|^p, & \text{if } v \geq 2, \end{cases}$$

where $\rho(\cdot)$ is a continuous function supported on $[-2,2]$. For example, $\rho(\cdot)$ is an infinitely differentiable function supported on $(-2,2)$ and equals to 1 on $[-1,1]$. For any $R > 0$, the family $\{f(\cdot,t)|t \in \mathbb{R}\}$ is equicontinuous on $[-R,R]$. However, there is also no a pointwise limit function of any sequence $\{f(\cdot, \cdot + t_n)\}$, as $t_n \to +\infty$. Hence, $f(v,t) \notin C^{\text{p.p.}}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$.

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